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Website: www.aarf.asia Email : editor@aarf.asia , editoraarf@gmail.com

EXACT DEFINITION OF MATHEMATICS

Dharmendra Kumar Yadav
Assistant Professor, Department of Mathematics
Shivaji College, University of Delhi, Raja Garden, Delhi-27

ABSTRACT

The aim of the article is to propound a simplest and exact definition of mathematics in a single sentence. It is observed that all mathematical and non-mathematical subjects whether science, arts, language or commerce, follow the same steps and roots to develop, they all consist of three parts: assumptions, properties and applications. These three terms make the exact definition of Mathematics, which can be applied to all subjects also. Therefore all subjects can be brought under the same umbrella of definition consisting of these three terms. Following this mathematics has been defined as the study of assumptions, its properties and applications. Then different branches of mathematics have been discussed. A short paragraph has been devoted to technical teachers and students on engineering mathematics. In last how should we teach mathematics has been emphasized? A special focus on the type of assignment has been mentioned. This article will be useful for mathematics teachers and its learners, if it is discussed in the first few lectures of undergraduate and post graduate level as well as will be more fruitful for technical students as they can understand and apply it better than non-technical students.

Key Words: Axiom, Theorem, Properties, Conjecture.

AMS Subject Classification: 97D30, 00A05, 00A06, 03E65, 01A80, 97D20

Introduction

Carl Friedrich Gauss referred mathematics as the queen of science but unfortunately students fear from this queen, although the subject is very essential to the growth of many other disciplines. The science of mathematics depends on the mental ability. It is the means to develop the thinking power and reasoning intelligence, which sharps the mind and makes it creative. The development of human beings and their culture depend on the development of mathematics. This is why, it is known as the base of human civilization. It is also the language of all material science and the centre of all engineering branches which revolve around it. Therefore it is the past, present and future of all sciences. ***Narlikar*** has focused on the importance of mathematics by mentioning that in 1957 when the Soviet Union launched the first satellite Sputnik, the United States realized that to match it, *the teaching of mathematics had to receive boost*. After that many major steps have been taken to improve the quality education of mathematics not only in USA but in the world too.

Previous Attempt to Define Mathematics

Although the research in mathematics has covered a milestone, a major drawback has been seen in the literature of mathematics that it could not be defined properly, so that all mathematical subjects can be combined in short and in a single sentence. No consensus on the definition of mathematics prevails, even among professionals. According to ***Wikipedia***, *mathematics has no generally accepted definition* and there is not even consensus on whether mathematics is an art or science. ***Gunter M. Ziegler*** mentions that in *German Wikipedia* the definition is interesting in a different way: it stresses that *there is no definition of mathematics, or at least no commonly accepted one*. Even the famous book by ***Richard Courant*** and ***Herbert Robbins*** entitled "What is Mathematics?" (and subtitled "An Elementary Approach to Ideas and Methods") does not give a satisfactory answer. He claims that *it is impossible to give a good definition in a sentence or two*. A great many professional mathematicians take no interest in a definition of mathematics, or consider it undefinable.

Traditionally *it is defined as the scientific study of quantities, including their relationship, operations and measurements expressed by numbers and symbols*. In mathematics dictionary by ***James & James*** *it has been defined as the science of logical study of numbers, shape, arrangement, quantity, measure and many related concepts*. Today it is usually described as a science that investigates abstract structures that it created itself for their properties and patterns". According to Wikipedia, '*Mathematics is the study of quantity, structure, space*. Mathematics seeks out patterns and uses them to formulate new conjectures. ***Aristotle*** has defined mathematics as '*The science of quantity*?'. ***Benjamin Pierce*** defined it as '*Mathematics is the science that draws necessary conclusions*'. ***Haskell Curry*** defined mathematics simply as "*the science of formal systems*". ***Albert Einstein*** stated that "*as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality*". More recently, ***Marcus du Sautoy*** has called mathematics "the Queen of Science …… the main driving force behind scientific discovery". Thus although all most all great mathematicians stated something for it, no generally accepted definition could be produced. A little attempt has been done in this article to define mathematics in a single sentence and exact form, which will be accepted for centuries without any counter example.

Basic Terms of Mathematics

In every mathematical subject we find some general terms like axioms, properties, theorem, etc. These are the basics of the subjects whose meanings are given as follows:

Axioms: *James* & James stated that the axioms of a subject are the basic propositions from which all other propositions can be derived. They are accepted as the starting points and are accepted true without any proof. With the help of axioms we decide whether a given mathematical statement is true or false. It is also known as assumptions or hypothesis or postulates or propositions.

Therefore mathematics can be regarded as a set and study of assumptions, because it starts with axiom. But it ends with reality. Here the statement 'end with reality' means although we start with assumptions, but after its application and in the final result, we reach at the real destination.

Example: One right angle is equal to 90^0 is an assumption. It has no proof. Although now there are many properties from which it can be proved, but the property from which it will be proved, will also be an assumption.

Property: Any mathematical statement derived using axioms is known as a property. A straight angle is equal to 180^0 is a property, which is proved as the sum of two right angles. It can be further divided in two parts: Theorem and Conjecture.

Theorem: A general conclusion which can be proved by the help of axioms is called a theorem. In general it is proved logically using assumptions as true.

Example: The sum of the three angles of a triangle is 180^0 is a theorem.

Conjecture: A mathematical statement which has many examples but cannot be proved or yet to be proved is known as conjecture. A well known conjecture is *Goldbach's Conjecture* which states that 'Every even integer greater than 4 can be written as sum of two odd primes'. So far either a proof or a counter example has not been found.

Applications: If we apply the assumptions and its properties to solve real life problems, we say that such type of assumptions have applications.

Example: The sum of the three angles of a triangle is 180^0 is an application of the assumption that one right angle is 90^0 and a straight angle is 180^0.

Thus we see that in mathematical subject we have three main terms: assumptions, properties and applications. So we can say that every mathematical subject is composed of three terms: assumptions, properties and applications. Thus we can define '***Mathematics is the study of assumptions, its related properties and applications***'. In fact every subject is the set of assumptions, its properties and applications as has been explained in the analysis below:

Analysis of Some Properties

Analysis 1: We remember that the sum of the three angles of a triangle is 180^0 as shown in fig.-1. Fig.-5 shows that one right angle is equal to 90^0. Fig.-3 and Fig.-4 show that two right angles make one straight angle which is equal to 180^0. Fig.-2 shows that the sum of three angles of a triangle is equal to 180^0, which is generally derived from the fact that when three angles <A, <B and <C are placed on a straight line with their vertices coinciding at one point, makes a straight angle which is equal to 180^0.

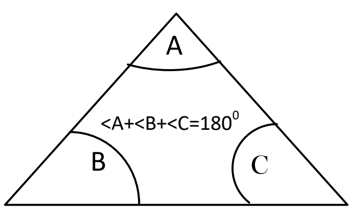


Fig.-1

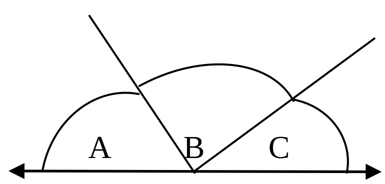


Fig.-2

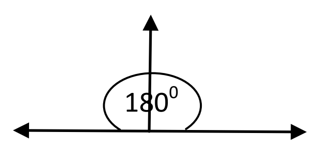


Fig.-3

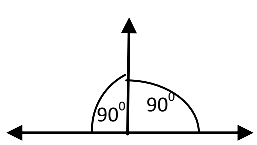


Fig.-4

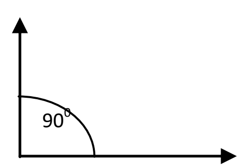


Fig.-5

We know that in geometry angles are measured in terms of right angle. In **British System**, a right angle is divided into 90 equal parts called Degrees, so a right angle makes 90^0. In **French System**, a right angle is divided into 100 equal parts called Grades, so a right angle makes 100^g. Whereas in **Circular Measure**, a right angle is equal to $\pi/2$. Thus the sum of the three angles of a triangle is 180^0 in British system, 200^g in French system and π radian in Circular measure system.

Therefore we first supposed that a right angle is equal to 90^0. Then we developed the definition of a linear pair of angles, for it the sum of two angles is 180^0 and then developed a property that a straight angle is equal to 180^0. By the help of this property, we proved that the sum of the three angles of a triangle is 180^0. What we followed here is that, first we assumed, then developed a property and finally applied it.

Analysis 2: A spherical balloon is pumped at the rate of 10 cubic inches per minute, find the rate of increase of its radius when its radius is 15 inches.

Let y be the volume and x the radius of the balloon at any time t. Then $\frac{dy}{dt}=10$ cubic inches per minute and we have to find $\frac{dx}{dt}$ when x=15 inches. Since the balloon is spherical, $y=\frac{4}{3}\pi r^3$.

$$\frac{dy}{dt}=4\pi x^2\frac{dx}{dt}\Rightarrow\frac{dx}{dt}=\frac{\frac{dy}{dt}}{4\pi x^2}=\frac{10}{4\pi x^2}=\frac{10}{4\pi 15^2}=\frac{1}{90\pi}$$

Hence rate of increase of the radius when radius is 15 inches is $\frac{1}{90\pi}$ inch/minute. For the function y=f(x), the differential coefficient of y with respect to x has been denoted and defined by

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

provided that the limit exists. It has been called the measurement of rate of change in y with respect to x. After that the derivative of $y=x^n$ is $\frac{dy}{dx}=nx^{n-1}$, was found with many other elementary functions. Then it was applied in many such problems as discussed above. So it also followed the three main steps: assumption, properties and applications. Similarly many more examples in mathematics can be analyzed.

Analysis 3: Let us consider the shadow formation on a screen by a point source of light. A source of light at S falls on the opaque body AB and makes a shadow A'B' on the screen.

Let us consider the following:

d_{sh}: diameter of the shadow;

d_b: diameter of the body

S: source of light;

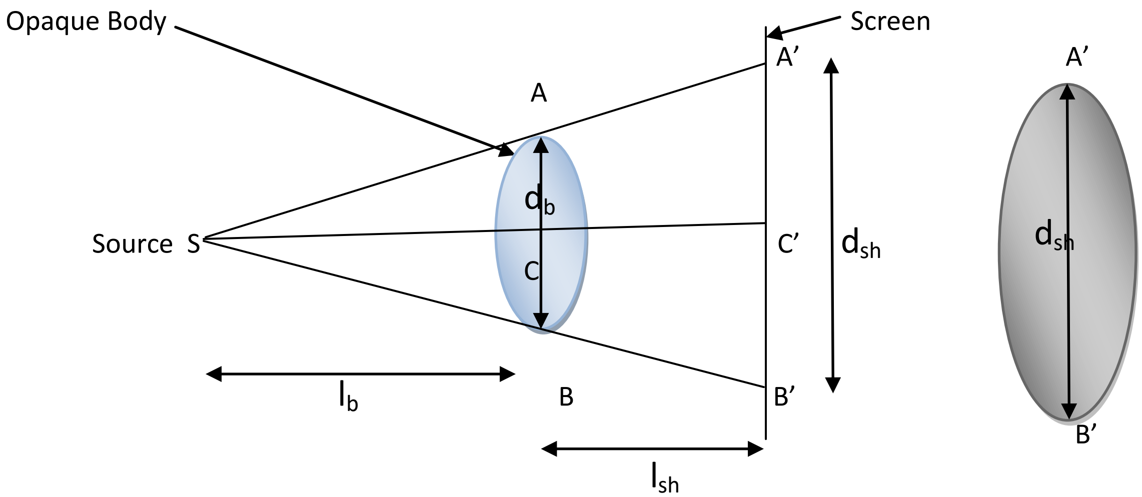
AB: Opaque Body;

A'B': shadow of the body on screen

l_b: distance between source & opaque body;

l_{sh}: distance between opaque body & screen.

The distances between different objects have been indicated in the following figure:



Now since triangles SAB and SA'B' are similar, we have

$$\frac{SC}{SC'} = \frac{AB}{A'B'} \Rightarrow \frac{l_b}{l_b + l_{sh}} = \frac{d_b}{d_{sh}} \Rightarrow d_{sh} = d_b\left(1 + \frac{l_{sh}}{l_b}\right)$$

From this we conclude that the size of the shadow is always greater than the size of the opaque body or object. It will be equal to the size of the object if the screen is in contact with the opaque body. As the screen is moved away from the opaque body the shadow size increases. To find the size of the shadow we used the concepts of similar triangles, which is itself an assumption and its properties.

Analysis 4: In language we first learn its alphabets, then words formed using alphabets and finally we make sentences using many words. For example, in English language, we first study from A to Z. Then we study words like Ram, Eat, Apple, etc having special meanings followed by grammar as properties. Finally we make sentence Ram eats an apple. So every language follows the three steps of development: assumption, properties and application. In English 26 alphabets are assumption, words and grammar are both assumptions and properties, where as a complete sentence is the application.

The same procedures follow in other languages like Hindi, Urdu, Sanskrit, etc. Similarly in Commerce, we assume the definitions of GDP, NDP, Income, Development, Growth, etc. and then apply it to study about the status of the society or nations.

From above examples, analysis and the basic terms of mathematical subjects, we can define mathematics in the shortest and compact form as '*Mathematics is the study of assumptions, its properties and applications*', which can be taken as the exact definition of mathematics. In fact '***all mathematical and non-mathematical subjects are the set and study of assumptions, its properties and applications***', whether it is science, arts, commerce, literature, etc.

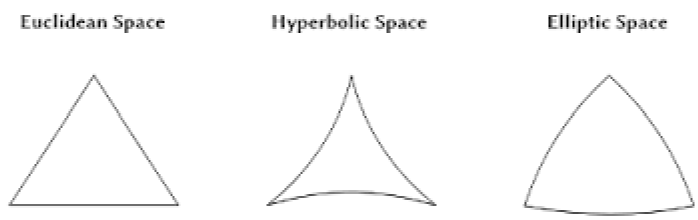
Branches of Mathematics

As far as the branch of mathematics is concerned, it is divided into four fields: Arithmetic, Algebra, Analysis and Geometry. In ***arithmetic*** we learn about numbers and basic arithmetical operations. When we apply arithmetic in solving real life problems, we get equations and thus lead to ***Algebra***. When the basic properties of arithmetic and algebra fail, we need analysis. In ***analysis*** we generally study about limits, continuity, etc.

In ***Geometry*** we study about shapes and size of the figures. It is divided into two parts: ***Plane (or Euclidean) Geometry*** and ***Spherical (or Non-Euclidean) Geometry***. Spherical geometry is further divided into two parts: ***Elliptical (or Riemann) Geometry*** and ***Non-elliptical (or Hyperbolic) Geometry***.

The basic difference among the above is that, in plane geometry, the sum of the three angles of a triangle if 180^0 where as in spherical geometry it is not equal to 180^0, but either greater than or less than of it. In elliptical geometry it is more than 180^0 but in hyperbolic geometry it is less than 180^0.

The shapes of a triangle in above three geometries are as follows:



Similarly other properties and figures can be studied in three geometries.

Pure and Applied Mathematics

For the sake of simplicity, mathematics is divided into two branches: Pure and Applied Mathematics.

Pure Mathematics is concerned with increasing knowledge of the subject rather than using knowledge in practical ways, i.e. its study is theoretical. For example, trigonometry, geometry, set theory, vector, etc. It is concerned with concepts and ideas that do not necessarily have any immediate practical application. Its importance can be better understood by the well known statement "***A pure mathematician makes dreams even beyond the imagination of human beings, and it is the scientists and technologists to apply them***."

Applied Mathematics is concerned with using knowledge of pure mathematics. It is not theoretical but practical. In it we use the theories and concepts of pure mathematics. For example, mechanics, dynamics, statics, physics, etc. It is concerned with the use of mathematical theories and principles as tools to solve problems in any field, whether it is research or problems in actuarial science, economics, financial analysis, market research analysis, meteorology, oceanography, aviation, aerodynamics, robotics, population studies, commercial surveys, physics, chemistry, biology, social sciences, earth sciences, industrial research and development.

The study of problems in applied mathematics leads to new developments in pure mathematics and theories developed in pure mathematics often find applications later. Work in applied mathematics requires a theoretical background, which enables the mathematician to understand the physical dimensions and technicalities of the problem. The study of pure and applied mathematics is interdependent. ***Applied mathematics is like a flowing river having pure mathematics on its two banks playing like domain on the boundaries***. The two branches are so interrelated and mixed that no sharp (or dividing) line can be drawn between them.

Engineering Mathematics

It is a part of applied mathematics. ***Kreyszig*** has mentioned that for the sake of deciding the depth of studying mathematics, while teaching mathematics to the engineering students, we should limit our knowledge of mathematics to the extent as far as the applications of the subject are concerned. Technical students need solid knowledge of basic principles, methods, and results, and a clear perception of what engineering mathematics is all about, in all three phases of solving problems related to real and physical world: Modeling, Solving and Interpreting.

How To Teach Mathematics?

Ronning has pointed out that mathematics is becoming more and more important in study. More and more decisions are made and actions are being taken on the basis of mathematical models. So the problem arises that, what should we emphasize when we teach mathematics? What kind of understanding do we want the students to develop? What kind of mathematics, and how much, do all students need to know?

A simple answer of the question is that every chapter must be divided into three parts: assumptions, properties and applications. When we start teaching, we must mention that what are the basic assumptions in the chapter keeping in view that definition of a term is itself an assumption. What can we obtain from the assumptions and in last how and where can we apply these concepts? Even students may be allowed to remember the definition as their own assumptions and then try to find out some properties related to them and the previous mathematical or non-mathematical knowledge they have. In this way the learner will enjoy the subject and they will improve their ability of mathematical power.

As far as the engineering mathematics is concerned, ***Kreyszig*** states that it would make no sense to overload students with all kinds of little things that might be of occasional use. Instead it is important that students become familiar with ways to think mathematically, recognize the need for applying mathematical methods to engineering problems, realize that mathematics is a systematic science built on relatively few basic concepts and involving powerful unifying principles, and get a firm grasp for the interrelation between theory, computing and experiment.

What should be the Format of Assignment?

Assignments play an important role in learning. In general it has been found that the teacher gives some examples as assignments to solve. In this manner students generally do not care about basic assumptions. They apply formula blindly and miss the basic concepts. They miss the game of enjoyment that how properties are developed using assumptions, which finally fails our aim of increasing mathematical ability in students. In fact the proper format of assignments must follow the three main terms of the subject: assumption, properties and applications.

In other words, students must be directed to do assignment in three steps: first they must define the basic terms of the chapter followed by the properties with proof. They must make a table of the formulae and then solve at least five examples on each formulae or properties. In this way students learn the definition, formulae and understand the basic structure of the chapter, which makes them perfect in application. Thus our motives become more and more successful in increasing the interest of mathematics among students.

Conclusion:

Finally we conclude the exact definition of mathematics as the study of assumptions, its properties and applications. In teaching we must maintain the order of assumptions, properties, and applications. This order must be maintained in assignments to get the desired aim of teaching mathematics. Similarly all non-mathematical subjects can be defined as the set of assumptions, its properties and applications.

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References

- Boaler J., The elephant in the Classroom: Helping Children Learn & Love Maths, Chapter-1, Souvenir Press, 2008
- Burton D. M., Elementary Number Theory, 7th Indian Edition, McGraw Hill Education (India) Pvt Ltd, New Delhi, 2012, pp. 50-53
- Courant R., Robbins H., What is Mathematics? An elementary approach to Ideas and Methods, 2nd Edition, Oxford University Press, New York, 1996
- Gakkhad S. C., Teaching of Mathematic, N. M. Prakashan, Chandigarh, 1991
- James & James, Mathematics Dictionary, 4th Edition, CBS Publishers & Distributors, India, 2001, pp.23, 169-170, 239, 387
- Kennedy L. M. & Tipps S., Guiding Children's Learning of Mathematics, 8th Edition, Wadsworth Publishing Company An International Thomson Publishing Company, 1997
- Kreyszig E., Advanced Engineering Mathematics, 8th Edition, John Wiley & Sons, 2005
- Loney S. L., Plane Trigonometry, Part-I, Metric Edition, A. K. Publications, Agra, 1990, pp. 1-17
- Narlikar J. V., Mathematics: The Queen of Sciences, Education Article, Manorama Year Book 2013, pp. 480-484

- Nelson D., The Penguin Dictionary of Mathematics, Penguin Books, 3rd Edition, 2003, pp. 21, 78, 147, 185-186, 271, 399
- Ronning F., Developing Knowledge in Mathematics by Generalising and Abstracting, Mathematics Newsletter, Ramanujan Mathematical Society, Vol.17 (4), March 2008, pp.109-118
- Schleicher D. & Lackmann M., An Invitation to Mathematics From Competition to Research (Preface: What is Mathematics? by Gunter M. Ziegler), Springer, XIV, pp.220, 2011
- Sinha K. C., A Text Book of Calculus, Students' Friends, Patna, 1994, pp. 236-237
- Vygodsky M., Mathematical Handbook: Elementary Mathematics, MIR Publishers, Moscow, 1987
- Ziegler G. M. & Loos A., Teaching and Learning "What is Mathematics", www.google.com
- https://en.wikipedia.org/wiki/Hyperbolic_geometry
- https://en.wikipedia.org/wiki/Elliptic_geometry
- https://en.wikipedia.org/wiki/Definitions_of_mathematics