International Research Journal of Natural and Applied Sciences
ISSN: (2349-4077)
Impact Factor- 5.46, Volume 5, Issue 3, March 2018
Website- www.aarf.asia, **Email** : editor@aarf.asia , editoraarf@gmail.com

FIXED POINT RESULTS IN FUZZY MENGER SPACE

Rashmi Pathak
Research Scholar, Department of Mathematics and Computer Science, U.T.D,
Rani Durgavati University Jabalpur (M. P.), India

Manoj Kumar Shukla
Department of Mathematics, Institute for Excellence in Higher Education,
Bhopal, (M.P.), India

ABSTRACT

Some fixed point theorems are proved in Fuzzy Menger Space for weak Commuting mappings.

Keywords: Fuzzy Menger Space, Weak Commuting mappings, Common fixed point.

AMS Subject Classification: 47H10 and 54H12.

1. INTRODUCTION

It was Menger [3] who introduced the notation of the probabilistic metric space almost 75 years back in 1942. The investigation of physical quantities and physiological thresholds are adopted through the probabilistic generalization of metric space. Schweizer and Sklar [4] studied this concept and then the important development of Menger space theory was due to Sehgal and Bharucha-Reid [5]. Sessa [6] introduced weakly commuting maps in metric spaces. So many recent works have been done in fuzzy and menger space Kutukcu et. al. [1] & [2], also Singh and Jain [8] did lot of work in this field. Very recently in 2012 the Fuzzy Probabilistic Metric Space is used by R. Shrivastav, V. Patel and V. B. Dhagat [7]. By using the work mentioned above we proved some fixed point results for Fuzzy Menger Space.

2. PRELIMINARIES

Definition 2.1 A fuzzy probabilistic metric space (FPM space) is an ordered pair (X, F_α) consisting of a nonempty set X and a mapping F_α from $X \times X$ into the collections of all fuzzy distribution functions $F_\alpha \in R$ for all $\alpha \in [0,1]$. For $x, y \in X$ we denote the fuzzy distribution function by $F_\alpha(x, y)$ and $F_\alpha(x, y)\ (u)$ is the value of $F_\alpha(x, y)$ at u in R.

The functions $F_\alpha(x, y)$ for all $\alpha.$ [0,1] assumed to satisfy the following conditions:

FPM(1) $F_\alpha(x, y)(u) = 1\ \ u > 0$ iff $x = y$,

FPM(2) $F_\alpha(x, y)(0) = 0\ x, y$ in X,

FPM(3) $F_\alpha(x, y) = F_\alpha(y, x)\ x, y$ in X,

FPM(4) If $F_\alpha(x, y)(u) = 1$ and $F_\alpha(y, z)(v) = 1$

$\Longrightarrow F_\alpha(x, z)(u + v) = 1\ \forall\ x, y, z \in X$ and $u, v > 0$.

Definition 2.2 A commutative, associative and non-decreasing mapping $t : [0,1] \times [0,1] \to [0,1]$ is a t −norm if and only if $t(a, 1) = a\ \ \forall\ a \in [0,1]$, $t(0,0) = 0$ and $t\ (c, d) \geq t(a, b)$ for $c \geq a, d \geq b$.

Definition 2.3 A Fuzzy Manger Space is a triplet (X, F_α, t), where (X, F_α) is a FPM-space, t is a t-norm and the generalized triangle inequality

$$F_\alpha(x, z)(u + v) \geq t\ (F_\alpha(x, z)(u) F_\alpha(y, z)(v))$$

holds for all x, y, z in $X\ and\ u, v > 0$ and $\alpha \in [0,1]$.

The concept of neighborhoods in Fuzzy Menger space is introduced as follows:

Definition 2.4 Let (X, F_α, t) be a fuzzy Menger space. If $x \in X, \varepsilon > 0, \alpha \in [0,1]$ and $\lambda \in (0,1)$ then (ε, λ) - neighborhood of x, denoted by $U_x\ (\varepsilon, \lambda)$, is defined by

$$U_x\ (\varepsilon, \lambda) = \{y \in X : F_\alpha(x, y)(\varepsilon) > (1 - \lambda)\}.$$

An (ε, λ)-topology in X is the topology induced by the family

$\{U_x\ (\varepsilon, \lambda): x \in X, \varepsilon > 0, \alpha \in [0,1]\ and\ \lambda \in (0,1)\}$ of neighborhoods.

Remark: If t is continuous, then Fuzzy Menger space (X, F_α, t) is a Housdroff space in (,)-topology.

Let (X, F_α, t) be a complete fuzzy menger space and $A \subset X$. Then A is a bounded set if

$$\lim_{u\to\infty} inf F_\alpha(x, y)(u) = 1 \qquad \text{for all} \quad x, y \in A.$$

Definition 2.5 A sequence $\{x_n\}$ in (X, F_α, t) is said to be convergent to a point x in X if for every > 0 and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $x_n \in U_x\ (\varepsilon, \lambda)\ \forall\ n \geq N$ or equivalently $F_\alpha\ (x_n, x;) > 1 - \lambda$ for all $n \geq N$ and $\alpha \in [0,1]$.

Definition 2.6 A sequence $\{x_n\}$ in (X, F_α, t) is said to be cauchy sequence if for every > 0 and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that

$$\text{for all } \alpha \in [0,1],\ F_\alpha\ (x_n, x_m;) > 1 - \lambda\ \forall\ n, m \geq N.$$

Definition 2.7 A Fuzzy Menger space (X, F_α, t) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X for all $\alpha \in [0,1]$.

Let $(X. M, *)$ is a fuzzy Menger metric space with the following condition (Fuzzy menger space 6).

(FMS-6) $\qquad lim_{t\to\infty} F_\alpha(x, y, t) = 1,\ \forall\ x, y \in X$

In fuzzy Menger space consider the Following lemmas

Lemma 1. [8] Let $\{x_n\}$ be a sequence in a Menger space $(X, F_\alpha, *)$ with continuous t-norm $*$ and $t * t \geq t$. If there exists a constant $k \in (0, 1)$ such that

$$F_{\alpha(x_n, x_{n+1})}(kt) \geq F_{\alpha(x_{n-1}, x_n)}(t) \text{ for all } t > 0 \text{ and } n = 1, 2, \ldots,$$

then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2. [8] Let $(X, F_\alpha, *)$ be a Menger space. If there exists $k \in (0, 1)$ such that $F_{\alpha(x,y)}(kt) \geq F_{\alpha(x,y)}(t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

3. Main Results:

THEOREM (3.1): Let $(X, F_\alpha, *)$ be a complete fuzzy menger metric space with the condition (FMS-6) and let S and T be continuous mappings of X , then S and T have a common fixed point in X if there exists continuous mappings A of X into $S(X) \cap T(X)$ which commute weakly with S and T and

$$F_\alpha(A_x, A_y, qt) \geq inf\{F_\alpha(T_y, A_y, t), F_\alpha(S_x, A_x, t), F_\alpha(S_x, T_y, t), F_\alpha(A_x, T_y, t), F_\alpha(S_x, A_y, t)\}$$

$\ldots\ldots\ldots\ldots\ldots\ldots..(3.1.1)$

for all $x, y\, X, t > 0$ and $0 < q < 1$. Then S, T and A have a unique common fixed point..

Proof: We define a sequence $\{x_n\}$ such that $A(x_{2n}) = S(x_{2n-1})$ and $A(x_{2n-1}) = T(x_{2n})$ for n = 1, 2,….. .We shall prove that $\{A(x_{2n})\}$ is a Cauchy sequence.

$$F_\alpha(Ax_{2n}, Ax_{2n+1}, qt) \geq$$
$$inf\{F_\alpha(Tx_{2n+1}, Ax_{2n+1}, t), F_\alpha(Sx_{2n}, Ax_{2n}, t), F_\alpha(Sx_{2n}, Tx_{2n+1}, t), F_\alpha(Ax_{2n}, Tx_{2n+1}, t), F_\alpha(Sx_{2n}, Ax_{2n+1}, t)\}$$

$$F_\alpha(Ax_{2n}, Ax_{2n+1}, qt) \geq$$
$$inf\{F_\alpha(Ax_{2n}, Ax_{2n+1}, t), F_\alpha(Ax_{2n+1}, Ax_{2n}, t), F_\alpha(Ax_{2n+1}, Ax_{2n}, t), F_\alpha(Ax_{2n}, Ax_{2n}, t), F_\alpha(Ax_{2n+1}, Ax_{2n+1}, t)\}$$

$$F_\alpha(Ax_{2n}, Ax_{2n+1}, qt) \geq inf\{F_\alpha(Ax_{2n}, Ax_{2n+1}, t), F_\alpha(Ax_{2n+1}, Ax_{2n}, t), F_\alpha(Ax_{2n+1}, Ax_{2n}, t), 1, 1\}$$

$$F_\alpha(Ax_{2n}, Ax_{2n+1}, qt) \geq$$
$$inf\{F_\alpha(Ax_{2n}, Ax_{2n+1}, \tfrac{t}{q}), F_\alpha(Ax_{2n+1}, Ax_{2n}, \tfrac{t}{q}), F_\alpha(Ax_{2n+1}, Ax_{2n}, \tfrac{t}{q}), 1, 1\}$$

$$\Longrightarrow F_\alpha(Ax_{2n}, Ax_{2n+1}, qt) \geq F_\alpha(Ax_{2n-1}, Ax_{2n}, \frac{t}{q})$$

By induction

$$F_\alpha(Ax_{2k}, Ax_{2m+1}, qt) \geq F_\alpha(Ax_{2m}, Ax_{2k-1}, \frac{t}{q})$$

For every k and m in N, Further if $2m + 1 > 2k$, then

$$F_\alpha(Ax_{2k}, Ax_{2m+1}, qt) \geq F_\alpha(Ax_{2m}, Ax_{2k-1}, \frac{t}{q}) \dots \dots \dots \geq F_\alpha(Ax_0, Ax_{2m+1}, \frac{t}{q^{2k}}) \dots \dots (3.1.1)(a)$$

If $2k > 2m + 1$, then

$$F_\alpha(Ax_{2k}, Ax_{2m+1}, qt) \geq F_\alpha(Ax_{2k-1}, Ax_{2m}, \frac{t}{q}) \dots \dots \dots \geq$$
$$F_\alpha(Ax_{2k-(2m+1)}, Ax_0, \frac{t}{q^{2k}}) \dots \dots \dots \dots . (3.1.1)(b)$$

By simple induction with $(3.1.1)(a)$ and $(3.1.1)(b)$ we have

$$F_\alpha(Ax_n, Ax_{n+p}, qt) \geq F_\alpha(Ax_0, Ax_p, \frac{t}{q^n}).$$

For $n = 2k, p = 2m + 1$ or $n = 2k + 1, p = 2m + 1$ and by (FPM-4)

$$F_\alpha(Ax_n, Ax_{n+p}, qt) \geq F_\alpha(Ax_0, Ax_1, \frac{t}{2q^n}) * F_\alpha(Ax_1, Ax_p, \frac{t}{q^n}) \dots \dots (3.1.1)(c)$$

If $n = 2k, p = 2m$ or $n = 2k + 1, p = 2m$

For every positive integer p and n in N, by nothing that

$$F_\alpha(Ax_0, Ax_p, \frac{t}{q^n}) \to 1 \text{ as } n \to \infty$$

Thus $\{Ax_n\}$ is a Cauchy sequence. Since the space X is complete $\exists\ z \in X$, such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Tx_{2n} = z$$

It follows that $Az = Sz = Tz$ and so

$$F_\alpha(Az, AAz, qt) \geq$$
$$\inf\{F_\alpha(TAz, AAz, t), F_\alpha(Sz, Az, t), F_\alpha(Sz, TAz, t), F_\alpha(Az, TAz, t), F_\alpha(Sz, AAz, t)\}$$

$$F_\alpha(Az, AAz, qt) \geq F_\alpha(Sz, TAz, t) \geq F_\alpha(Az, AAz, t) \dots \dots \dots \geq F_\alpha(Az, AAz, \frac{t}{q^n})$$

Since $\lim_{n\to\infty} F_\alpha(Az, AAz, \frac{t}{q^n}) = 1$, so $Az = AAz$.

Thus z is common fixed point of A, S and T.

For **uniqueness**, let v $(v \neq z)$ be another common fixed point of S, T and A.

By (*3.1.1*) we write

$$F_\alpha(Az, Av, qt) \geq$$
$$inf \{F_\alpha(Tv, Av, t), F_\alpha(Sz, Az, t), F_\alpha(Az, Tv, t), F_\alpha(Sz, Av, t)\}$$

$$F_\alpha(Az, Av, qt) \geq inf \{F_\alpha(v, v, t), F_\alpha(z, z, t), F_\alpha(z, v, t), F_\alpha(z, v, t)\}$$

$$F_\alpha(Az, Av, qt) \geq F_\alpha(z, v, t)$$

This implies that $F_\alpha(z, v, qt) \geq F_\alpha(z, v, t)$

Therefore **by lemma 2,** we write $z = v$.

THEOREM (3.2): Let $(X, F_\alpha, *)$ be a complete fuzzy menger metric space with the condition (FM-6) and let S and T be continuous mappings of X in X, then S and T have a common fixed point in X if there exists continuous mappings A of X into $S(X) \cap T(X)$ which commute weakly with S and T and

$$F_\alpha(Ax, Ay, qt) \geq$$
$$inf \{F_\alpha(Tx, Ay, t), F_\alpha(Sx, Ax, t), F_\alpha(Sx, Ty, t), \frac{F_\alpha(Sx, Ty, t)}{F_\alpha(Ax, Ty, t)}, \frac{F_\alpha(Ty, Ay, t)}{F_\alpha(Sx, Ax, t)}, \frac{F_\alpha(Sx, Ax, t)}{F_\alpha(Ty, Ay, t)}\}$$
$$\ldots\ldots\ldots (3.2.1)$$

for all $x, y \in X, t > 0$, and $0 < q < 1$. Then S, T and A have a unique common fixed point.

Proof: We define a sequence $\{x_n\}$ such that

$Ax_{2n} = Sx_{2n-1}$ and $Ax_{2n-1} = Tx_{2n}$ $n = 1,2, \ldots\ldots\ldots$

We shall prove that $\{Ax_n\}$ is a Cauchy sequence. By (*3.2.1*), we have

$$F_{\alpha}(Ax_{2n}, Ax_{2n+1}, qt) \geq$$
$$\inf\ \{F_{\alpha}(Tx_{2n+1}, Ax_{2n+1}, t), F_{\alpha}(Sx_{2n}, Ax_{2n}, t), F_{\alpha}(Sx_{2n}, Tx_{2n+1}, t), \frac{F_{\alpha}(Sx_{2n}, Tx_{2n+1}, t)}{F_{\alpha}(Ax_{2n}, Tx_{2n+1}, t)}, \frac{F_{\alpha}(Tx_{2n+1}, Ax}{F_{\alpha}(Sx_{2n}, Ax}$$

$$F_{\alpha}(Ax_{2n}, Ax_{2n+1}, qt) \geq$$
$$\inf\ \{F_{\alpha}(Ax_{2n}, Ax_{2n+1}, t), F_{\alpha}(Ax_{2n+1}, Ax_{2n}, t), F_{\alpha}(Ax_{2n+1}, Ax_{2n}, t), \frac{F_{\alpha}(Ax_{2n+1}, Ax_{2n}, t)}{F_{\alpha}(Ax_{2n}, Ax_{2n}, t)}, \frac{F_{\alpha}(Ax_{2n}, Ax_{2}}{F_{\alpha}(Ax_{2n+1}, Ax}$$

$$\Rightarrow F_{\alpha}(Ax_{2n}, Ax_{2n+1}, qt) \geq$$
$$\inf\ \{F_{\alpha}(Ax_{2n}, Ax_{2n+1}, t), F_{\alpha}(Ax_{2n+1}, Ax_{2n}, t), F_{\alpha}(Ax_{2n+1}, Ax_{2n}, t), 1, 1, 1\}$$

$$\Rightarrow F_{\alpha}(Ax_{2n}, Ax_{2n+1}, qt) \geq F_{\alpha}(Ax_{2n-1}, Ax_{2n}, \frac{t}{q})$$

By induction

$$F_{\alpha}(Ax_{2k}, Ax_{2m+1}, qt) \geq F_{\alpha}(Ax_{2m-1}, Ax_{2k-1}, \frac{t}{q})$$

For every k and m in N, Further if $2m+1 > 2k$, then

$$F_{\alpha}(Ax_{2k}, Ax_{2m+1}, qt) \geq F_{\alpha}(Ax_{2k-1}, Ax_{2m}, \frac{t}{q}) \ldots\ldots\ldots \geq$$
$$F_{\alpha}(Ax_{0}, Ax_{2m+1-2k}, \frac{t}{q^{2k}}) \ldots\ldots\ldots\ldots (3.2.1)(a)$$

If $2k > 2m+1$, then

$$F_{\alpha}(Ax_{2k}, Ax_{2m+1}, qt) \geq$$
$$F_{\alpha}(Ax_{2k-1}, Ax_{2m}, \frac{t}{q}) \ldots\ldots\ldots\ F_{\alpha}(, Ax_{2k-(2m+1)}, Ax_{0}, \frac{t}{q^{2k}}) \ldots\ldots\ldots.. (3.2.1)(b)$$

By simple induction with $(3.2.1)(a)$ and $(3.2.1)(b)$

We have

$$F_{\alpha}(Ax_{n}, Ax_{n+p}, qt) \geq F_{\alpha}(Ax_{0}, Ax_{p}, \frac{t}{q^{n}})$$

For $n = 2k, p = 2m+1$ or $n = 2k+1, p = 2m+1$ and by (FPM-4)

$$F_{\alpha}(Ax_{n}, Ax_{n+p}, qt) \geq F_{\alpha}(Ax_{0}, Ax_{1}, \frac{t}{2q^{n}}) * M(Ax_{1}, Ax_{P}, \frac{t}{q^{n}}) \ldots\ldots (3.2.1)(c)$$

If $n = 2k, p = 2m$ or $n = 2k+1, p = 2m$

For every positive integer p and n in N, by nothing that

$F_\alpha(Ax_0, Ax_p, \frac{t}{q^n}) \to 1$ as $n \to \infty$

Thus $\{Ax_n\}$ is a Cauchy sequence.

Since the space X is complete there exists $z \in X$, such that

$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_{2n-1} = lim\ Tx_{2n}{}_{n\to\infty} = z$

It follows that = = and so

$F_\alpha(Az, AAz, qt) \geq$

$inf\ \{F_\alpha(TAz, AAz, t), F_\alpha(Sz, Az, t), F_\alpha(Sz, TAz, t), \frac{F_\alpha(Sz, Az, t)}{F_\alpha(Az, TAz, t)}, \frac{F_\alpha(TAz, AAz, t)}{F_\alpha(Sz, Az, t)}, \frac{F_\alpha(Sz, Az, t)}{F_\alpha(TAz, AAz, t)}\}$

$F_\alpha(Az, AAz, qt) \geq F_\alpha(Sz, TAz, t) \geq F_\alpha(Az, AAz, t) \geq \cdots \ldots \ldots \geq$
$F_\alpha(Az, AAz, \frac{t}{q^n})$

Since $lim_{n\to\infty} F_\alpha(Az, AAz, \frac{t}{q^n}) = 1$

$\Longrightarrow Az = AAz$

Thus z is common fixed point of A, S and T.

For uniqueness, let v $(v \neq z)$ be another common fixed point of S, T and A.

By (3.2.1), we write

$F_\alpha(Az, Av, qt) \geq$

$inf\ \{F_\alpha(Tv, Av, t), F_\alpha(Sz, Az, t), F_\alpha(Sz, Tv, t), \frac{F_\alpha(Sz, Tv, t)}{F_\alpha(Az, Tv, t)}, \frac{F_\alpha(Tv, Av, t)}{F_\alpha(Sz, Az, t)}, \frac{F_\alpha(Sz, Az, t)}{F_\alpha(Tv, Av, t)}\}$

$F_\alpha(Az, Av, qt) \geq F_\alpha(z, v, t)$

This implies that

$$F_\alpha(z, v, qt) \geq F_\alpha(z, v, t)$$

Therefore by **lemma 2**, we write $z = v$.

This completes the proof of the Theorem

REFERENCES

[1]. Kutukcu S, Sharma S and Tokgoz, H. "A Fixed Point Theorem in Fuzzy metric Spaces"Int. Journal of Math. Analysis, Vol. 1, 2007, no. 18, 861– 872

[2]. Kutukcu, S. Tuna, A and Yakut, A. T. Generalized contraction mapping principle in intuitionistic Menger spaces and an application to differential equations, Appl. Math.Mech. **28** (2007), no. 6, 779-809.

[3]. Menger, K. Statistical metric, Proc. Nat. Acad. **28** (1942), 535-537.

[4]. Schweizer B and Sklar,A. Probabilistic Metric Spaces, North-Holland, Amsterdam,1983.

[5]. Sehgal V. M. and Bharucha-Reid, A.T. Fixed point of contraction mapping on PM spaces, Math. Systems Theory **6** (1972), 97-100

[6]. Sessa, S. On weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. Beograd **32** (1982), 149-153

[7]. Shrivastav, R., Patel V and Dhagat, V.B. "Fixed point theorem in fuzzy menger spaces satisfying occasionally weakly compatible mappings "Int. J. of Math. Sci. & Engg. Appls. , Vol.6 No.VI, 2012, 243-250.

[8]. Singh B. and Jain,S. A fixed point theorem in Menger space through weak compatibility. Math. Anal. Appl. **301** (2005), 439-448