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STUDY OF CHARACTERIZATION OF SEPARATION AXIOMS IN BITOPOLOGICAL SPACES

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ABSTRACT

We analyse properties of some existing classes of non-continuous functions using the properties derived by Kariofills as well as some bitopological separation axioms, and conditions for its regularity and normality based on multivalued and single-valued functions. J. E. Kelly introduced the notion of different separation axioms for non-continuous function which have been further developed by Kariofillis on bitopological spaces by taking different separation axioms using a set of non-continuous functions introducing the concept of ij-θ-closure operator.

Keywords:- bitopological space, pairwise Hausdroff, homeomorphism, disjoint subsets, regularity, normality.

Introduction

We state some basic notions of bitopological space and relevant definitions and properties. Let us denote a bitopological space (X,Q_1,Q_2) by X. For a subset A of (X, Q_1, Q_2), Q_i-int(A) and Q_i *-c1(A)* stand for the interior and closure of A in (X,Q_i) respectively where i = 1,2. A point $x \in X$, is said to be in the ij-θ-closure of a subset A of X, $x \in ij\text{-}\theta\text{-}cl(A)$, iff for every Q_i-open set U containing x, $Q_j\text{-}cl(U) \cap A \neq \phi$ where $i,j = 1,2$, $i \neq j$. $A(\subset X)$ is called ij-θ-closed iff $A = ij\text{-}\theta\text{-}cl(A)$. A space (X, Q_1,Q_2) is called pair-wise R_1 iff for any two points z , $y \in X$, such that $x \notin$. $Q_i\text{-}cl\ \{y\}$, there is a Q_i-open set U and *a* Q_j-open set V such that $x \in U$, $Y \in V$ and $U \cap V = \phi$. The space X is called pairwise Hausdroff, iff for any two distinct points X, y of X, there exist a Q_i-open neighbourhood U of x and a Q_j-open nbd V of y such that $U \cap V = \phi$ (resp. $Q_j\text{-}cl(U) \cap Q_i\text{-}cl(V) = \phi$), where i,j == 1,2 and $i \neq j$. X is called pairwise regular, iff for each point x of X and each Q_i-closed set F such that $x \notin F$, there exist a Q_i-open set U and a Q_j-open set V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$, for i, j == 1 and 2, and i ≠ j. X will be called pairwise normal, iff for each Q_1-closed set A and a Q_2-closed set B disjoint from A, there exist a Q_1-open set V and a Q_2-open set U such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$. (i, j = 1, 2 and i ≠ j). *A* mapping $f: (X, Q_l, Q_2) \rightarrow (Y, P_l, P_2)$ is said to be pairwise strongly θ-continuous weakly continuous) iff for each $x \in X$ and each P_i-*open* nbd U of $f(x)$, there exists a Q_i-open nbd V of x such that $f(Q_j\text{-}cl(V)) \subset U$ (resp. $f(Q_j\text{-}cl(V)) \subset P_j\text{-}cl(U)$, $J(V) \subset P_j\text{-}cl(U)$).

Theorem

A function f : X, Q_1, Q_2) → (Y, P_1, P_2) *is* pairwise weakly continuous iff $f(Q_i\text{-cl}(A)) \subset$ ij-θ-cl(f(A)), for each $A \subset Y$.

Proof

Case-i) Let us assume f to be pairwise weakly continuous and let $Y \in f(Q_i\text{-}cl(A))$. There exists $x \in X$ such that $x \in Q_i\text{-}cl(A)$ and $f(x) = y$. Let V be a P_i-open nbd of $f(x)$ Y. Since the function f is pairwise weakly continuous, there exists $U_i \in Q_1$ with $x \in u_i$, such that $f(U_i) \subset P_j\text{-}cl(V)$. Now, $x \in Q_i\text{-}cl(A) =\} U_i, \cap A \neq \phi \Longrightarrow f(U_i) \cap f(A) \neq \phi \Longrightarrow P_j\text{-}cl(V) \cap f(A) \neq \phi \Longrightarrow f(x) \in ij\text{-}\theta\text{-}cl(f(A))$, i.e., $y \in ij\text{-}\theta\text{-}cl(f(A))$. Hence $f(Q_i\text{-}c1(A)) \subset ij\text{-}\theta\text{-}cl(f(A))$.

Case-ii) Only if, conversely, let $x \in X$ be arbitrary and V be a P_i-open nbd of $f(x)$. Then $f(x) \notin ij\text{-}\theta\text{-}cl(Y - P_j\text{-}cl(V))$ and hence $f(x) \notin ij\text{-}\theta\text{-}cl[f f^{-1} (Y - P_j\text{-}cl(V))]$. By hypothesis of theorem (1.3) $f(x) \notin f(Q_i\text{-}cl(f^{-1}(Y - P_j\text{-}cl(V))))$ so that $x \notin Q_i\text{-}cl(X - f^{-l}(P_j\text{-}cl(V)))$. Thus there exists a Q_i-open nbd U of x such that $U \subset f^{-l}(P_j\text{-}cl(V))$ and hence $f(U) \subset P_j\text{-}cl(V)$. Thus f is pairwise weakly continuous.

Theorem

If f, g: f: $(X, Q_l, Q_2) \rightarrow (Y, P_l, P_2)$ are pairwise almost continuous functions then the set $A = \{a \in X: f(a) \in ij\text{-}\theta\text{-}cl\{g(a)\}\}$ *is* Q_i-closed in X and the set $B = \{(a, b) \in X \times X : f(a) \in ij\text{-}\theta\text{-}cl\{g(b)\}\}$ is T_i-closed in $(X \times X, T_1, T_2)$, where $i: = Q_i \times Q_j$, *for* $i,j = 1,2$, $i \neq$ j.

Proof

Case-I: we first show that B is T_i-closed in $X \times X$. $(a, b) \notin B \Rightarrow f(a)$ (j. $ij\text{-}\theta\text{-}cl\{g(b)\} \Longrightarrow$ there exists $V_i \in P_i$ with $f(a) \in V_i$ and $W_j \in P_j$ with $g(b) \in W_j$ such that $V_i \cap W_j = \phi$, which shows that $V_i \cap W_j = \phi$ implies $P_i\text{-}int(P_j\text{-}cl(V_i)) \cap P_j\text{-}int(P_i\text{-}cl(W_j)) = \phi$. Since *f, g* are pairwise almost continuous, it implies $f(G_i) \subset P_i\text{-int}\ (P_j\text{-}cl(V_i))$ and $g(H_j) \subset P_j\text{-}int(P_i\text{-}cl(W_j))$ for some Q_i-open nbd G_i of *a* and some Q_j-open nbd *H* , of b, Then $(G_i \times H_j) \cap B = \phi$, where $(a, b) \in G_i \times H_j$. Let us now suppose $(x, y \in (G_i \times Hj) \cap B$. Then $f(x) \in f(G_i) \subset P_i\text{-}int(P_j\text{-}cl(V_i)) \in P_i$ and $g(y) \in g(H_j) \subset P_j$ - $int(P_i\text{-}cl(W_j)) \in P_j$. Also since $(x, y) \in B$, $f(x) \in ij\text{-}\theta\text{-}cl\{g(y)\}$ so that $P_i\text{-}int(P_j\text{-}cl(V_i)) \cap P_j\text{-int}\ (P_i\text{-}cl(W_j) \neq \phi$ which is a contradiction. But $(G_i \times H_j) \cap B = \phi$, $(a,b) \notin Ti\text{-}cl(B)$. Thus *B* is T_i-closed in $X \times X$. *w*e now show that the restriction to $\Delta(= \{(a, a) : a \in X\})$ of the i[th] projection mapping $P_i : (X \times X,T_1,) \rightarrow (X,Q_i)$ is a homeomorphism (for i = 1,2). Let $a \in A$, then $f(a) \in ij\text{-}\theta\text{-}cl\{g(a)\}$. This implies $(a, a) \in B \cap \Delta$ and hence $a \in P_i(B \cap \Delta)$. Since *B* is T_i-closed in $X \times X$, $B \cap \Delta$ is closed in the relative topology of Δ. Thus $P_i(B \cap \Delta)$ is Q_i-closed in *X* Hence, the theorem is proved. We state the results derived by S.K. Sen on characterization of bitopological separation axioms.

Theorem

Let f, g $(X, Q_l, Q_2) \rightarrow (Y, P_l, P_2)$ *be pairwise* θ*-continuous functions. If* (Y, P_l, P_2) *is pairwise Urysohn then the set* $\{a \in X: f(a) = g(a)\}$ *is ij-*θ-closed in X and the set $\{(a, b) \in X \times X : f(a) = g(b)\}$ *is ij-*θ*-closed in* $(X \times X,T_1,T_2)$ *(where* $T_i = Q_i$ X Q_j,. *i,j* = 1, 2; i ≠ *j).*

Proof

Let $B = \{(a,b) \in X \times X: f(a) = g(b)\}$. If $(a, b) \in X \times X - B$, we have $f(a) \neq g(b)$. Since *Y* is pairwise Urysohn, there exist a P_i-open nbd V_I of $f(a)$ and a P_j-open nbd V_2 of $g(b)$ such that $P_j\text{-}cl(V_1) \cap P_i\text{-}c1(V_2) = \phi$. Since *f* and *g* are pairwise θ-continuous there exist a Q_i-open nbd U_1 of *a* and a Q_j-open nbd U_2 of *b* such that $f(Q_j\text{-}cl(U_1)) \subset P_j\text{-}cl(V_i)$ and $g(Q_i\text{-}cl(U_2)) \subset P_i\text{-}cl(V_2)$. Then $f(Q_j\text{-}cl(U_1)) \cap g(Q_i\text{-}cl(U_2)) = \phi$ we thus obtain that $(Q_j\text{-}cl(U_1) \times Qi\text{-}cl(U_2)) \cap B = \phi$. Thus $[T_j\text{-}cl(U_1 \times U_2)] \cap B = \phi$. Such that $(a,b) \notin ij\text{-}\theta\text{-}cl(B)$. Hence *B* becomes ij-$\theta$-closed in $(X \times X, T_1, T_2)$. we now establish that the set $\{a \in X: f(a) = g(a)\}$ is ij- θ -closed in *X*. By using the some examples, it is shown that in the converses are not true in general.

Theorem

The property of being a pairwise Q* $T_{12/3}$ is a topological invariant.

Proof

Let B be a σ_i - Q* compact subset of (Y, σ_1, σ_2). Let h: (X, τ_1, τ_2) → (Y, σ_1, σ_2) be a pairwise Q* homeomorphism. Then h^{-1}(B) is a τ_i - Q* compact subset of (X, τ_1, τ_2). Put A = h^{-1} (B). But (X, τ_1, τ_2) is pairwise Q* $T_{12/3}$.

Accordingly, A is τ_j - Q* closed. But, then h (A) is σ_j - Q* closed because h is a τ_j - Q* closed map. That is, h (h^{-1} (B)) = B. Hence B is σ_2 - Q* closed. Consequently, (Y, σ_1, σ_2) is pairwise Q* $T_{12/3}$.

Theorem

Every pairwise Q* $TT_{1\,1/2}$ space is pairwise Q* T_0.

Proof

Suppose that X is pairwise Q*$T_{1\,1/2}$ space. Let x ≠ y in X. Then there exists a τ_i - Q* open set U and τ_j - Q* open set V such that U ∩ V = ϕ and x ∈ U, y ∈ V or x ∈ V, y ∈ U.

⇒ X is pairwise Q* T_0

Definition

A bitopological space X is said to be a pairwise Q$\boldsymbol{T_{2\,1/2}}$ space if x and y are distinct points in X then there exist a τ_i - Q* open neighborhood U of x and τ_j - Q* open neighborhood V of y such that τ_j - Q* cl (U) ∩ τ_i - Q* cl (V) = ϕ, where i, j = 1, 2 and i ≠ j.

Theorem

The property of being a pairwise Q*$T_{2\,1/2}$ space is hereditary.

Proof

Let X be a pairwise Q*$T_{2\,1/2}$ space. Let Y be a subspace of X, Let x, y ∈ Y with x ≠ y. Then x ≠ y in X. But X is Pairwise Q*$T_{2\,1/2}$ space,

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⇒ there exist a τ_i- Q* open neighborhood U of X and τ_j- Q* open neighborhood V of Y such that U ∩ V = ϕ. But Y ⊂ X.

⇒ there exist a τ_i-Q* open neighborhood $\bar{U}$ of X and τ_j - Q* open neighbourhood $\bar{V}$ of Y such that $\bar{U} \cap \bar{V}$ = ϕ. Hence Y is a pairwise Q*$T_{2\ 1/2}$ space.

Theorem

(i) Every pairwise Q*$T_{2\ 1/2}$ space is pairwise Q*T_2.

(ii) Every pairwise Q*$T_{2\ 1/2}$ space is pairwise T2 ½ space.

(i) Definition

A Q* T_1 - space X is said to be a pairwise Q*$\boldsymbol{T_{44\ 11/22}}$ space if for each τ_i- Q* closed set A and τ_j - Q* closed set B with A ∩ B = ϕ there exists a τ_i- Q* open set V ⊃ B and a τ_j- Q* open set U ⊃ A such that τ_i - Q*cl (U) ∩τ_j - Q* cl (V) = ϕ, where i, j = 1, 2 and i ≠ j.

(ii) Definition

A Q* T_1 - space X is said to be a pairwise Q*$\boldsymbol{T_{51/2}}$ space if for every subsets A and B of X such that τ_i- Q* cl (A) ∩ B = ϕ and A ∩τ_j - Q* cl (B) = ϕ there exists a τ_j - Q* open set U & a τ_i- Q* open set V such that A ⊂ U and B ⊂ V, τ_i- Q* cl (U) ∩τ_j- Q* cl (V) = ϕ, where i , j = 1 , 2 and i ≠ j.

Theorem

Every pairwise Q*$T_{5\ 1/2}$ space is pairwise Q*T_5 space.

Proof

Let X be a pairwise Q*$T_{5\ 1/2}$ space. Then X is Q*T_1 - space. Let A and B are disjoint subsets of X. Then A = τ_i- Q* cl (A) and B = τ_j- Q* cl (B).

⇒ τ_i- Q* cl (A) ∩ B = A ∩ B = ϕ and A ∩τ_j - Q* cl (B) = A ∩ B=ϕ.

Hence, A and B are Q* separated sets in X.

Since X is pairwise Q*$T_{5\ 1/2}$space, we have τ_i - Q* open set U and τ_j- Q* open set V such that A ⊂ U and B ⊂V, τ_i - cl (U) ∩τ_j - cl (V) = ϕ.

$$\Rightarrow U \cap V = \phi.$$

Theorem

(i) Every pairwise Q*$T_{5\ 1/2}$ space is pairwise $T_{5\ 1/2}$ space.

(ii) Every pairwise Q*$T_{5\ 1/2}$ space is pairwise Q*$T_{4\ 1/2}$ space.

Proof

Let X be a pairwise Q*$T_{5\ 1/2}$ space. Then X is Q* T_1 - space. Let A and B are disjoint closed subsets of X.

Then A = τ_i - Q* cl (A) and B = τ_j - Q* cl (B).

$\Rightarrow \tau_i$ - Q* cl (A)∩B = A∩B = ϕ and A ∩τ_j - Q* cl (B) = A∩B =ϕ.

Hence A and B are Q*separated sets in X. But X is pairwise Q*$T_{5\ 1/2}$ space, then there exists a τ_i - Q* open set V ⊃ B and τ_i - Q*open set U ⊃ A such that τ_i - Q* cl (U) ∩ τ_j - Q* cl (V) = ϕ.

$$\Rightarrow U \cap V = \phi.$$

Therefore, pairwise Q*$T_{5\ 1/2}$ space is pairwise Q*$_4$½ space. Hence, the theorem is proved.

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