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Operational matrix for Atangana–Baleanu derivative based on Genocchi polynomials for solve fractional differential equations

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Abstract

Fractional calculus, a generalization of classical calculus dealing with derivatives and integrals of arbitrary orders, has emerged as a powerful tool for modeling complex phenomena in various fields of science and engineering. Fractional differential equations (FDEs), which incorporate fractional derivatives, often provide more accurate and realistic descriptions of real-world processes exhibiting memory effects and non-local interactions compared to their integer-order counterparts. However, finding analytical solutions to FDEs is often challenging, necessitating the development of robust numerical techniques. One such approach involves the use of operational matrices based on orthogonal polynomials. These matrices transform the fractional differential and integral operators into algebraic matrices, thereby reducing the FDE to a system of algebraic equations that can be solved numerically. Among the various definitions of fractional derivatives, the Atangana–Baleanu (AB) derivative has gained significant attention due to its non-singular and non-local kernel based on the Mittag-Leffler function. This kernel avoids the singularities present in some other fractional derivative definitions, making it suitable for modeling certain physical problems more effectively.

Keywords:

Operational, matrix, derivative, polynomials, fractional, differential, equations

Introduction

Genocchi polynomials, a sequence of polynomials with interesting number-theoretic properties, have also found applications in numerical analysis. They are defined by the generating function: et+12text=n=0∑∞Gn(x)n!tn where Gn(x) denotes the n-th Genocchi polynomial. These polynomials possess several useful properties, including recurrence relations, differentiation and integration formulas, which make them suitable as basis functions for approximating solutions of differential equations. (Yildirim, 2022)

The development of an operational matrix for the Atangana–Baleanu derivative based on Genocchi polynomials provides a novel and potentially efficient numerical method for solving FDEs involving this particular fractional derivative. The key idea is to express the solution of the FDE as a linear combination of Genocchi polynomials. By utilizing the properties of Genocchi polynomials and the definition of the Atangana–Baleanu derivative, one can derive an operational matrix that represents the action of the fractional derivative on the vector of Genocchi polynomials.

The unknown solution of the FDE is approximated by a finite series of Genocchi polynomials: y(x)≈i=0∑NciGi(x)=cTG(x) where c=[c0,c1,...,cN]T is the vector of unknown coefficients and G(x)=[G0(x),G1(x),...,GN(x)]T is the vector of Genocchi polynomials. The Atangana–Baleanu derivative of the basis vector G(x) is expressed in terms of the same basis vector using an operational matrix DAB(α): ABDαG(x)≈DAB(α)G(x) The entries of the operational matrix DAB(α) are determined based on the properties of the Atangana–Baleanu derivative and the Genocchi polynomials. This often involves applying the fractional derivative to each Genocchi polynomial and then expressing the result as a linear combination of the same Genocchi polynomials. By substituting the polynomial approximation and the operational matrix into the given FDE, the differential equation is transformed into a system of algebraic equations in terms of the unknown coefficients c. The resulting algebraic system, along with any initial or boundary conditions, can be solved using standard numerical techniques to determine the values of the coefficients ci. Once the coefficients are found, the approximate solution of the FDE is obtained by substituting these coefficients back into the polynomial approximation. (Baskonus, 2020)

$$\left(I_{0,x}^{\mu,v,\eta}\, t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)\,\Gamma(\lambda - v + \eta)}{\Gamma(\lambda - v)\,\Gamma(\lambda + \mu + \eta)} x^{\lambda - v - 1} \quad (\lambda > 0, \lambda - v + \eta > 0)$$

............e.q. ..1

$$\left(J_{x,\infty}^{\mu,v,\eta}\, t^{\lambda-1}\right)(x) = \frac{\Gamma(v-\lambda+1)\,\Gamma(\eta-\lambda+1)}{\Gamma(1-\lambda)\,\Gamma(v+\mu-\lambda+\eta+1)} x^{\lambda-v-1} \quad (v-\lambda+1>0, \eta-\lambda+1>0)$$

.............e.q 2

The operational matrix method based on Genocchi polynomials for the Atangana–Baleanu derivative offers several potential advantages. It utilizes the well-behaved non-singular kernel of the Atangana–Baleanu derivative. Genocchi polynomials form a complete basis, allowing for the approximation of sufficiently smooth functions. It reduces the FDE to a system of algebraic equations, which are generally easier to solve numerically. With a sufficient number of basis polynomials, the method can achieve a good level of accuracy.

The derivation of the operational matrix for the Atangana–Baleanu derivative with respect to Genocchi polynomials can be complex. Solving the resulting algebraic system, especially for a large number of basis polynomials, can be computationally expensive. A rigorous analysis of the convergence and stability of the method is crucial but can be challenging.

The Atangana–Baleanu (AB) derivative is a novel fractional derivative introduced by Abdon Atangana and Dumitru Baleanu in 2016. It utilizes the generalized Mittag-Leffler function as its kernel, which is non-local and non-singular, addressing some limitations of the classical Riemann-Liouville and Caputo fractional derivatives that employ a power-law kernel. The AB derivative has gained significant attention due to its effectiveness in modeling various real-world phenomena exhibiting non-exponential decay or memory effects.

One of the crucial aspects of working with fractional derivatives, particularly for solving fractional differential equations (FDEs) numerically, is the development of operational matrices. An operational matrix is a matrix representation of a linear operator, such as a derivative or integral, with respect to a chosen basis of functions. When applied to the vector of coefficients of a function expanded in that basis, the operational matrix yields the vector of coefficients of the transformed function (e.g., its derivative or integral) (Baleanu, 2021)

Literature Review

Fabrizio et al. (2020): Common choices include polynomial bases (e.g., Legendre, Chebyshev, power basis) or other orthogonal functions. The choice of basis often depends on the problem being solved and the desired accuracy. This step involves applying the definition of the ABC or ABR derivative to each ϕj(t) in the chosen basis. This often requires careful evaluation of the integral involving the Mittag-Leffler function. The derivative aABDtαϕj(t) needs to be expanded as a linear combination of the basis functions {ϕ0(t),ϕ1(t),...,ϕN(t)}. The coefficients of this expansion form the j-th column of the operational matrix D(α).

Roslan et al. (2021): The explicit form of the operational matrix will vary depending on the chosen basis functions and whether the Caputo or Riemann-Liouville version of the AB derivative is used. For example, if a power basis {1,t,t2,...,tN} is used, applying the AB derivative to each tj will yield a result that needs to be expressed as a linear combination of the same power basis. This often involves the properties of the Mittag-Leffler function and can be complex.

Khan et al. (2023): If orthogonal polynomials like Legendre or Chebyshev polynomials are used, the AB derivative of these polynomials needs to be projected back onto the space spanned by these polynomials to determine the entries of the operational matrix.

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Once the operational matrix of the Atangana–Baleanu derivative is constructed, it can be used to numerically solve fractional differential equations involving this derivative. By discretizing the time domain and using the operational matrix to represent the AB derivative, FDEs can be transformed into systems of algebraic equations, which can then be solved using standard numerical techniques.

$$F_1[a,b,b';c;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} b_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad \max\{|x|,|y|\} < 1;$$

.........e.q. 3

$$F_2[a,b,b';c,c';x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} b_m (b')_n}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| + |y| < 1;$$

..............e.q. .4

$$F_3[a,a',b,b';c;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad \max\{|x|,|y|\} < 1;$$

.........e.q. 5

$$F_4[a,b;c,c';x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad \sqrt{|x|} + \sqrt{|y|} < 1$$

.........e.q. ..6

The properties of the operational matrix can provide insights into the stability of solutions of fractional-order systems modeled with the AB derivative. Operational matrices can be used in the analysis and design of controllers for systems described by fractional-order differential equations with the AB derivative. The non-local nature of the AB derivative makes it suitable for certain image processing tasks, and operational matrices can facilitate the numerical implementation of fractional-order filters. The Mittag-Leffler kernel of the AB derivative is well-suited for modeling processes that exhibit non-exponential decay, and operational matrix methods provide a way to obtain numerical solutions for such models.

$$F[f(x);\xi] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx,$$

.....e.q. 7

$$(1-t)^{-1} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x) t^n.$$

.............e.q. 8

Constructing the operational matrix for the Atangana–Baleanu derivative can be challenging due to the complexity of the Mittag-Leffler function and the need to express the derivative of the basis functions in terms of the original basis. Numerical integration or approximation techniques may be required to evaluate the entries of the matrix.

The choice of basis functions significantly affects the accuracy and efficiency of the method. The convergence properties of the chosen basis and the size of the operational matrix (which depends on the number of basis functions used) need to be carefully considered.

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \, {}_1F_1[-n;\ 1+\alpha;\ x], \ Re(\alpha) > -1,$$

................e.q. 9

$$L_n^{(\alpha)}(x) \ = \ \lim_{|\beta|\to\infty} \left\{ P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right) \right\}.$$

............e.q. 10

The operational matrix of the Atangana–Baleanu derivative provides a powerful tool for the numerical analysis and solution of fractional differential equations involving this novel derivative. By representing the AB derivative in a matrix form with respect to a chosen basis, complex FDEs can be transformed into algebraic systems that are amenable to numerical solutions. While the construction of these matrices can be intricate, their application opens up new avenues for modeling and analyzing systems with non-local and non-singular memory effects in various fields of science and engineering. Ongoing research continues to explore efficient methods for constructing these operational matrices for different basis functions and to further investigate their properties and applications.

Conclusion

The development of an operational matrix for the Atangana–Baleanu derivative based on Genocchi polynomials represents a promising approach for obtaining numerical solutions to fractional differential equations involving this specific type of fractional derivative. Further research into the properties of this operational matrix, its implementation for various types of FDEs, and the analysis of its accuracy and efficiency will be valuable in establishing its practical utility in solving real-world problems modeled by fractional calculus.

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